3) Eigenmodal solutions

1. Eigenvalue equation

Eigenmodal solutions = solutions of Maxwell's equations in source-free regions, time harmonic $e^{j\omega t}$

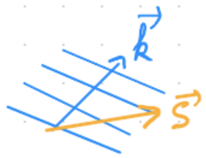
$$\begin{cases}\vec\nabla\times\vec E = -j\omega(\hat\mu\vec H+\hat\beta\vec E)\\ \vec\nabla\times\vec H = j\omega(\hat\varepsilon\vec E+\hat\alpha\vec H)\end{cases}\qquad \begin{cases}\vec\nabla\cdot\vec B=0\\ \vec\nabla\cdot\vec D=0\end{cases}$$

We assume the solutions are proportional to $e^{-j\vec k\cdot\vec r}$ (plane waves)

$$\vec\nabla \leftrightarrow -j\vec k$$

$$\begin{cases}-j\vec k\times\vec E = -j\omega(\hat\mu\vec H+\hat\beta\vec E)\\ -j\vec k\times\vec H = j\omega(\hat\varepsilon\vec E+\hat\alpha\vec H)\end{cases}\qquad \begin{cases}-j\vec k\cdot\vec B=0\\ -j\vec k\cdot\vec D=0\end{cases}$$

$\Rightarrow$ $\vec k\perp\vec D$, $\vec k\perp\vec B$, but not necessarily to $\vec E$ & $\vec H$ $\Rightarrow$ $\vec S=\frac{1}{2}\vec E\times\vec H^*$ not necessarily parallel to $\vec k$



$$\vec k\times\vec E = \begin{pmatrix}k_yE_z-k_zE_y\\ k_zE_x-k_xE_z\\ k_xE_y-k_yE_x\end{pmatrix} = \begin{pmatrix}0 & -k_z & k_y\\ k_z & 0 & -k_x\\ -k_y & k_x & 0\end{pmatrix}\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix} = \hat K\vec E$$

$\hat K$ is not invertible

$$\begin{cases}-j\hat K\vec E = -j\omega(\hat\mu\vec H+\hat\beta\vec E)\\ -\cancel{j}\hat K\vec H = \cancel{j}\omega(\hat\varepsilon\vec E+\hat\alpha\vec H)\end{cases}\Rightarrow\begin{cases}\vec H = \hat\mu^{-1}(\omega^{-1}\hat K\vec E-\hat\beta\vec E) & (1)\\ -\hat K\vec H = \omega\hat\varepsilon\vec E+\omega\hat\alpha\vec H & (2)\end{cases}$$

Plug (1) in (2) $\Rightarrow -\hat K\hat\mu^{-1}\omega^{-1}\hat K\vec E+\hat K\hat\mu^{-1}\hat\beta\vec E = \omega\hat\varepsilon\vec E+\cancel{\omega}\hat\alpha\hat\mu^{-1}\cancel{\omega^{-1}}\hat K\vec E-\omega\hat\alpha\hat\mu^{-1}\hat\beta\vec E$

$\times(-\omega)$ $\Rightarrow \hat K\hat\mu^{-1}\hat K\vec E+\omega(\hat\alpha\hat\mu^{-1}\hat K\vec E-\hat K\hat\mu^{-1}\hat\beta)\vec E = \omega^2(\hat\alpha\hat\mu^{-1}\hat\beta-\hat\varepsilon)\vec E$

$\Rightarrow$ $\underline{\hat M\vec E=0}$ with $\boxed{\hat M = \hat K\hat\mu^{-1}\hat K+\omega(\hat\alpha\hat\mu^{-1}\hat K-\hat K\hat\mu^{-1}\hat\beta)-\omega^2(\hat\alpha\hat\mu^{-1}\hat\beta-\hat\varepsilon)}$

2. Resolution strategy

$$\hat{M}\vec{E} = \vec{0}$$

If $\det \hat{M} \neq 0$, $\vec{E} = \hat{M}^{-1}\cdot\vec{0} = \vec{0}$ $\Rightarrow$ trivial solution, happens whatever $\vec{k}$ is

If $\underbrace{\det \hat{M} = 0 = f(\omega, k)}_{\text{dispersion relation}}$ $\Rightarrow$ $\underbrace{\vec{E} \neq \vec{0}}_{\text{eigenmode}}$ belongs to the kernel (null space) of $\hat{M}$

3. Test on isotropic nonchiral media

$$\left.\begin{array}{l} \hat{\alpha} = \hat{\beta} = \hat{0} \\ \hat{\varepsilon} = \varepsilon \mathbb{1}_3,\ \hat{\mu} = \mu \mathbb{1}_3 \end{array}\right\} \Rightarrow \left[\hat{K}^2 + \omega^2 \mu \varepsilon \mathbb{1}_3\right]\vec{E} = \vec{0} \quad \text{with } \hat{K}^2 = \hat{K}\cdot\hat{K} = \begin{pmatrix} -k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_x k_y & -k_x^2 - k_z^2 & k_y k_z \\ k_x k_z & k_y k_z & -k_x^2 - k_y^2 \end{pmatrix}$$

Noting $k_0^2 = \omega^2 \mu \varepsilon$, we get:

$$\begin{pmatrix} k_0^2 - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_x k_y & k_0^2 - k_x^2 - k_z^2 & k_y k_z \\ k_x k_z & k_y k_z & k_0^2 - k_x^2 - k_y^2 \end{pmatrix}\vec{E} = \vec{0}$$

Let us assume $\vec{k} \parallel \vec{x}$: $\vec{k} = [k_x, 0, 0]^T$ (mere redefinit° of coord syst)

$$\Rightarrow \hat{M} = \begin{pmatrix} k_0^2 & 0 & 0 \\ 0 & k_0^2 - k_x^2 & 0 \\ 0 & 0 & k_0^2 - k_x^2 \end{pmatrix} \qquad \det M = 0 \Rightarrow \boxed{k_x = \pm\sqrt{k_0} = \pm\,\omega\sqrt{\mu\varepsilon}}$$

Under this condition $\hat{M}\vec{E} = 0$ becomes $\begin{pmatrix} k_0^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\vec{E} = 0 \Rightarrow \boxed{\vec{E} = \begin{pmatrix} 0 \\ E_y \\ E_z \end{pmatrix}}$

We retrieve the known fact that in isotropic materials, $\vec{E} \perp \vec{k}$.

4. Propagation in chiral media

$$\begin{cases} \vec{D} = \varepsilon \vec{E} \overset{\alpha}{-j\gamma} \vec{H} \\ \vec{B} = \mu \vec{H} \overset{\beta}{+j\gamma} \vec{E} \end{cases}$$ asuming a lossless reciprocal chiral medium $(\varepsilon, \mu, \gamma \in \mathbb{R})$

$$\hat{M}\vec{E} = \vec{0} \quad \text{with} \quad \hat{M} = \hat{K}^2 + \omega \underbrace{(\alpha - \beta)}_{-2j\gamma} \hat{K} + \omega^2 (\mu\varepsilon - \underbrace{\alpha\beta}_{\gamma^2})$$

Since the material is isotropic, we can pick $\vec{k} = k_z \hat{z}$ for example

$$\Rightarrow \hat{M} = \begin{pmatrix} \omega^2(\mu\varepsilon - \gamma^2) - k_z^2 & 2j\omega\gamma k_z & 0 \\ -2j\omega\gamma k_z & \omega^2(\mu\varepsilon - \gamma^2) - k_z^2 & 0 \\ 0 & 0 & \omega^2(\mu\varepsilon - \gamma^2) \end{pmatrix}$$

$\det \hat{M} = 0 \Rightarrow \left[\omega^2(\mu\varepsilon - \gamma^2) - k_z^2\right]^2 - 4\omega^2\gamma^2 k_z^2 = 0$ (assuming $\mu\varepsilon - \gamma^2 \neq 0$)

$$\left(\omega^2(\mu\varepsilon - \gamma^2) - k_z^2 + 2\gamma\omega k_z\right)\left(\omega^2(\mu\varepsilon - \gamma^2) - k_z^2 - 2\gamma\omega k_z\right) = 0$$

$\Rightarrow k_z^2 \mp 2\gamma\omega k_z - \omega^2(\mu\varepsilon - \gamma^2) = 0$ (2 possibilities)

$\Rightarrow k_z = \frac{1}{\cancel{2}}\left(\pm \cancel{2}\gamma\omega \pm \sqrt{\cancel{4}\omega^2\gamma^2 + \cancel{4}\omega^2(\mu\varepsilon - \gamma^2)}\right) = \pm\gamma\omega \pm \omega\sqrt{\varepsilon\mu}$ (2×2 solutions)

$\Rightarrow \boxed{k_z = \pm\omega\left(\sqrt{\varepsilon\mu} \pm \gamma\right)}$ 4 solutions

For $\boxed{k_z > 0}$ (+ sign) , we can look for the eigenmode plugging the DR in $\hat{M}.\vec{E} = \vec{0}$

$$\left[\omega^2(\mu\varepsilon - \gamma^2) - \omega^2(\sqrt{\mu\varepsilon} \pm \gamma)^2\right] E_x + 2j\omega\gamma(\omega\sqrt{\mu\varepsilon} \pm \gamma) E_y = 0$$

$$\Rightarrow \left(\cancel{\omega^2\mu\varepsilon} - \omega^2\gamma^2 - \cancel{\omega^2\mu\varepsilon} - \omega^2\gamma^2 \mp 2\omega^2\gamma\sqrt{\mu\varepsilon}\right) E_x + \left(2j\omega^2\gamma\sqrt{\mu\varepsilon} \pm 2j\omega^2\gamma^2\right) E_y = 0$$

$$\Rightarrow \left(-2\omega^2\gamma^2 \mp 2\omega^2\gamma\sqrt{\mu\varepsilon}\right) E_x + j\left(2\omega^2\gamma\sqrt{\mu\varepsilon} \pm 2\omega^2\gamma^2\right) E_y = 0$$

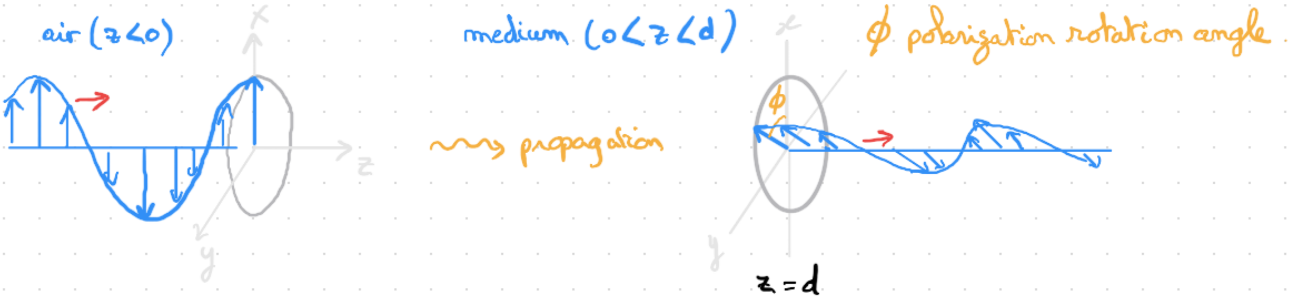
$$\Rightarrow E_x = +j \frac{2\omega^2\gamma\sqrt{\mu\varepsilon} \pm 2\omega^2\gamma^2}{\underbrace{\mp 2\omega^2\gamma\sqrt{\mu\varepsilon} - 2\omega^2\gamma^2}_{\mp}} E_y \Rightarrow \boxed{E_x = + \mp j E_y} \quad \begin{matrix}\text{LHCP} \\ \text{RHCP}\end{matrix} \; !$$

For $\boxed{k_z < 0}$ $\quad E_x = - \mp j E_y = \pm j E_y \quad \begin{matrix}\text{LHCP} \\ \text{RHCP}\end{matrix}$

Conclusion : A LHCP that travels with a phase velocity $v_{p_+} = (\sqrt{\mu\varepsilon} + \gamma)^{-1}$
and a RHCP that travels with a phase velocity $v_{p_-} = (\sqrt{\mu\varepsilon} - \gamma)^{-1}$
regardless of the propagation direction $\pm k_z$ = reciprocity.

5. Optical activity of chiral media

Optical activity: rotation of the polarisation of a linearly polarized wave as it propagates thru the medium



Why? LP is a sum of RHCP & LHCP which aquire different phases over d.

Calculation of ϕ:

$$\vec{E}_{in} = E_0 \vec{x}\, e^{-jk_0 z} = \frac{E_0}{2}\left[\overbrace{(\vec{x}+j\vec{y})}^{\text{LHCP}} + \overbrace{(\vec{x}-j\vec{y})}^{\text{RHCP}}\right] e^{-jk_0 z} \quad \text{for } z<0$$

Neglecting reflection: $\vec{E}_t = \frac{E_0}{2}\left[(\vec{x}+j\vec{y})\,e^{-j\beta_{LH}z} + (\vec{x}-j\vec{y})\,e^{-j\beta_{RH}z}\right]$

$$\Rightarrow \vec{E}_t = \frac{E_0}{2}\, e^{-j\frac{\beta_{RH}+\beta_{LH}}{2}z}\left[(\vec{x}+j\vec{y})\,e^{-j\frac{\beta_{LH}}{2}z}\,e^{j\frac{\beta_{RH}}{2}z} + (\vec{x}-j\vec{y})\,e^{-j\frac{\beta_{RH}}{2}}\,e^{+j\frac{\beta_{LH}}{2}z}\right]$$

$$= \frac{E_0}{2}\, e^{-j\beta_{AV}z}\left[\vec{x}\left(e^{-j\frac{\beta_{LH}-\beta_{RH}}{2}z} + e^{j\frac{\beta_{LH}-\beta_{RH}}{2}z}\right) + j\vec{y}\left(e^{-j\frac{\beta_{LH}-\beta_{RH}}{2}z} - e^{j\frac{\beta_{LH}-\beta_{RH}}{2}z}\right)\right.$$

$j\hat{y} \times 2j\sin\left(-\frac{1}{2}(\beta_{LH}-\beta_{RH})\right)$

$$= E_0\, e^{-j\beta_{AV}z}\left[\vec{x}\cos\left(\frac{\beta_{LH}-\beta_{RH}}{2}z\right) + \vec{y}\sin\left(\frac{\beta_{LH}-\beta_{RH}}{2}z\right)\right]$$

$$\tan\phi = \left.\frac{E_y}{E_x}\right|_{z=d} = \tan\left(\frac{\beta_{LH}-\beta_{RH}}{2}d\right) \Rightarrow \boxed{\phi = \frac{\beta_{LH}-\beta_{RH}}{2}d}$$

Here $\beta_{LH} = \omega\sqrt{\mu\varepsilon} + \gamma$, $\beta_{RH} = \omega\sqrt{\mu\varepsilon} - \gamma \Rightarrow \boxed{\phi = \gamma d}$

Rq: Optical activity is a reciprocal pol. rotation effect $\neq$ Faraday rotation (non-rec)

6. Power flow in chiral media

<u>Magnetic field</u> Assume $k_z = \omega(\sqrt{\mu\varepsilon} \pm \gamma)$ $E_x = \mp j E_y$ $\vec{\nabla} \times \vec{E} = -j\omega(\mu\vec{H} + j\gamma\vec{E})$

$$\Rightarrow \cancel{-j}\, k_z \vec{z} \times \vec{E} = \cancel{-j}\omega(\mu\vec{H} + j\gamma\vec{E}) \Rightarrow \omega\mu\vec{H} + j\omega\gamma\vec{E} = k_z \vec{z} \times \vec{E}$$

mode amplitude

$$\Rightarrow \vec{H} = \frac{1}{\omega\mu}\left[k_z \vec{z} \times \vec{E} - j\omega\gamma\vec{E}\right) \quad \text{with } \vec{E} = E_x(\vec{x} \pm j\vec{y}) \equiv E_0(\vec{x} \pm j\vec{y})$$

$$\Rightarrow \vec{H} = \frac{1}{\omega\mu}\left[k_z E_0(\vec{y} \mp j\vec{x}) - j\omega\gamma E_0(\vec{x} \pm j\vec{y})\right]$$

$$= \frac{E_0}{\omega\mu}\left[\omega\sqrt{\mu\varepsilon}(\vec{y} \mp j\vec{x}) \underbrace{\pm\, \omega\gamma(\vec{y} \mp j\vec{x})}_{+\omega\gamma(j\vec{x} \pm \vec{y}) = +j\omega\gamma(\cancel{\vec{x}} \mp j\vec{y})} - j\omega\gamma(\cancel{\vec{x}} \pm j\vec{y})\right]$$

$$\vec{H} = \frac{E_0}{\cancel{\omega}\mu}\left[\cancel{\omega}\sqrt{\mu\varepsilon}(\vec{y} \mp j\vec{x}) \pm 2\cancel{\omega}\gamma\vec{y}\right] = \frac{E_0}{\eta_0}\underbrace{(\vec{y} \mp j\vec{x})}_{\substack{\text{RHCP}\\ \text{LHCP}}} \pm \frac{2\gamma E_0}{\mu}\vec{y} \rightarrow \text{elliptical but TEM}$$

$$\vec{S} = \frac{1}{2}(\vec{E} \times \vec{H}^*) = \frac{|E_0|^2}{2\eta_0}\begin{pmatrix}1\\ \pm j\\ 0\end{pmatrix} \times \begin{pmatrix}\pm j\\ 1 \pm \frac{2\gamma\eta_0}{\mu}\\ 0\end{pmatrix} = \frac{|E_0|^2}{2\eta_0}\left(2 \pm \frac{2\gamma\eta_0}{\mu}\right)\vec{z}$$

$$\Rightarrow \boxed{\vec{S} = \frac{|E_0|^2}{\eta_0}\left(1 \pm \frac{\gamma}{\sqrt{\mu\varepsilon}}\right)\vec{z}}$$

Rq: 1) $\vec{S}(\gamma \to 0) = \frac{|E_0|^2}{\eta_0} = 2 \times \frac{|E_0|^2}{2\eta_0}$ expected.

$\vec{k} = \omega\sqrt{\mu\varepsilon}\left(1 \pm \frac{\gamma}{\sqrt{\mu\varepsilon}}\right)\vec{z}$, $\vec{S}\cdot\vec{k} > 0$ always, $\vec{S} \parallel \vec{k}$